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Towards a Complex Variable Interpretation of Peirce's Existential Graphs

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1. Background

Peirce's existential graphs were introduced in the period 1890–1910, as alternative forms of logical analysis. The three basic characteristics of the graphs lie in their specific ways to transfer logical information: *diagrammatic*, *intensional*, *continuous*. Peirce's systems of existential graphs contrast thus with the usual logical systems, whose algebraic, extensional and discrete emphases are closer to the specific conceptual combinatorics of set theory.

For almost a century, Peirce's graphs were forgotten, and only began to be studied again by philosophy Ph.D. students oriented towards logical research. Roberts (1963) and Zeman (1964) showed in their dissertations that Peirce's systems served as complete axiomatizations for well-known logical calculi: Alpha equivalent to classical propositional calculus, Beta equivalent to first-order logic on a purely relational language, fragments of Gamma equivalent to propositional modal calculi between S4 and S5. On another path, Burch (1991) showed that a relational fragment of Beta represented a genuine intensional logico-topological calculus, not reducible to an extensional relational calculus inside set theory.

The "standard" presentations of the graphs (see for example Roberts, 1973; Shin, 2002) have underlined the visual interest of the systems, but have just emphasized their originality as a diagrammatic *language*. The combinatorial syntax of the graphs has been described recursively in

Shin (2002), and a classical semantical interpretation has been proposed in Hammer (1998). On the other hand, the topological potentialities of the graphs were explored in Kauffman (2001), and their ties with game theory were studied in Pietarinen (2006).

Nonetheless, the profound connections of the graphs with *central areas in mathematics* have just beginning to be unravelled, thanks to two groundbreaking papers by Geraldine Brady and Todd Trimble. Inserting the graphs in the context of *monoidal categories*,[1] Brady & Trimble (2000a) have showed that *(a)* every Alpha graph gives rise to an algebraic operation in an algebraic theory in the sense of Lawvere (a particular case of a monoidal category); *(b)* Alpha's deduction rules can be obtained through factorization of "forces".[2] These are important results which open the way to a new combinatorial handling of the graphs, profiting from advanced techniques in categorical logic (Borceux, 1994; Jacobs, 1999). On the other hand, Brady & Trimble (2000b) have indicated how to represent Beta graphs by means of a category-theoretic relational calculus associated to a first-order theory. The representation does not use Freyd's allegorical calculus, nor Lawvere's hyperdoctrines, but a medium-complexity representational calculus, with "logical functors" which create quantifiers and which verify a "Beck-Chevalley" condition.[3] The free relational category which allows the representation is put in correspondence with a (monoidal) category of chord diagrams, using ideas from Joyal & Street (1991).

In spite of the preceding work, the advances obtained in a mathematical understanding of Peirce's graphs do not contemplate two of their main features: **intensionality** and **continuity**. The combination of the *intensional*

[1] Monoidal categories are categories equipped with a tensor functor, thanks to which a natural notion of abstract monoid can be defined. Monoidal categories are ubiquitous: cartesian categories (in particular, the category of sets), free word-category over *any* category, endofunctors category over *any* category, category of R-modules over a commutative ring R, etc. The abstract monoids definable in the monoidal category incarnate in the usual monoids, triples (or monads), R-algebras, etc.

[2] Given a monoidal category $\mathbf{C}$ with tensor product $\otimes$, and given a contravariant functor $F\colon \mathbf{C} \to \mathbf{C}$, a force for F is a natural transformation $\theta_{ab}\colon F(a) \otimes b \to F(a \otimes b)$. The forces, introduced by Max Kelly in the 1980's to solve difficult coherence problems (reduction of the commutativity of an infinity of diagrams to the commutativity of finite of them), have emerged afterwards in domains farther apart: curvatures in grassmannians, sub-riemannian geometry, weak forces in subatomic physics, counting operators in linear logic, etc. Here, the forces appear in another unexpected context: intuitionistic logic and existential graphs.

[3] Beck-Chevalley is a categorical expression of the syntactical idea that substitution of bound variables does not affect a logical formula. It corresponds to type uniformization in rewriting rules, and it also appears in mathematics around ideas of uniformization in some classes of algebras (algebraic groups: Chevalley; algebraic functors: Beck).

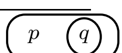
character of the graphs, expressed by the fact that an Alpha cut around *p* is just the opposite[4] of a Venn (extensional) diagram around *p*, and of its *topological* character, expressed through the continuity calculus (iterations and deiterations) of the line of identity, show that Peirce's graphs should be closer to a logic akin to intensional and topological considerations, that is, in fact, closer to intuitionistic logic (on this, the forthcoming papers Oostra (2011) and Zalamea (2011) should produce new lights).

On the other hand, if one considers the continuous plane where evolves Beta information, and the book of sheets where may evolve Gamma information, one can see that Peirce's graphs may be treated with some tools of complex analysis: *residues* for Alpha, *analytic continuation* for Beta, *Riemann surfaces* for Gamma. Moreover, since one of the natural models for the *logic of sheaves* (see Caicedo, 1995) is the fibration of germs of analytic functions, a natural connection between existential graphs and the logic of sheaves should emerge. In this case, Peirce's existential graphs would enter into the very core of mathematical knowledge, and could even help to understand the elusive "logic of complex variables".[5]

In the remaining parts of this paper, we provide some advances, but mostly guesses and conjectures, around what has been announced in our last paragraph.

2. A complex variable interpretation of Peirce's graph

The usual model for Peirce's Alpha and Beta graphs is imagined as a "sheet of assertion", where, on one hand (Alpha), nested cuts are marked in order to represent combinations of classical propositional formulas built on negation ("cut") and conjunction ("juxtaposition"), and, on the other hand (Beta), continuous lines are marked in order to represent existential quantification. In this *modus* of representation, a *superposition* between continuity (Beta) and discontinuity (Alpha) is *essential* for the good sake of the calculus: while Alpha is restricted to non over crossing, nested diagrams, Beta requires *crossing*, *not nested* diagrammatical procedures to obtain its full capability of representation. In fact, all the power of the *line of identity*

[4] If a graph of the form 

represents in Peirce's view the implication $p \to q$, the Venn extensional reading of the diagram produces exactly the opposite inclusion (implication): $\{x : q\} \subseteq \{x : p\}$.

[5] The *stability* of the complex additive-multiplicative plane (C, +, . , 0, 1) has been a source of many developments in model theory. On the other hand, the *instability* of the complex exponential is now under careful study (Zilber) and may hold some of the profound secrets of the logic of complex variables.

(the continuous line which represents existential quantification) rests in its ability to cross Alpha cuts, since the iterations and deiterations of the line correspond *precisely* to the use of *normal forms* in the associated underlying first-order logic.

This Beta superposition or crossing has not been sufficiently considered in its full, central, importance for the machinery of the graphs. If one would take it seriously, it would lead to *(i)* a study of the forms of continuity (Beta) and discontinuity (Alpha) present on the sheet of assertion, and, more significantly, to *(ii)* a study of forms of logical transfers (possibilities of proofs) and obstructions (impossibilities of proofs) as forms of *topological transfers (continuations) and obstructions (singularities)*. In a sense, this would lead then to a sort of homological bridging between the logical and the topological, but in an *inverse way* to the direction that usual homological machineries are produced.[6]

If the sheet of assertion is viewed as an infinite plane, the usual understanding of the sheet identifies it with the cartesian plane $\mathbf{R}^2$, an identification which helps to understand program *(i)* just indicated. But then, the topological transformations of the plane $\mathbf{R}^2$, viewed as transformations of two real variables, are highly artificial, fortuitous, hazardous – far from being "tame" in Grothendieck's sense –, and it would be very astonishing that the natural, universal, structural, logical *calculi* encoded in the graphs could be surfacing in some real variable calculations. Another *completely different* perspective is obtained if we view the infinite sheet of assertion as the **complex plane** $\mathbf{C}$. If, extensionally, looking just as sets, the cartesian plane $\mathbf{R}^2$ and the complex plane $\mathbf{C}$ are identical, *intensionally*, with their extremely different *calculi* of real variables and complex variables, the two planes differ in profound ways. Since the graphs do involve intensional logical information, it seems from the outset that the distinction may be fruitful. We will see that many additional technical ingredients support this view.

Program *(ii)* may in fact be well-founded on the theory of functions of complex variable. The analytical (also called holomorphic) functions can be described both locally (through power series expansions: Weierstrass)

[6] A *homology* is usually constructed to be able to understand the topological through the algebraic: *given* a topological space, a homology for the space is a chain of abelian groups which captures parts of the continuous information (deformations) of the initial space. A *cohomology* of the space is a homology with the order of the chain reversed, which gives rise to easier constructions in the chain: pullbacks, products, etc. In our proposal, instead of evolving *from* topological data, we would be going *towards* the topological, profiting in first instance from the known logic behaviour of the graphs (Roberts, 1963; Zeman, 1964).

and globally (through elliptic partial differential equations: Cauchy, Riemann), and the *good solidarity* between the local and the global explains their excellent (not artificial, not fortuitous, not hazardous) behaviour. Many technical accomplishments in the theory express this solidarity. A qualitative breakthrough is obtained when we observe, not only the *recto* of the **analytical functions**, but also its *verso*: the **meromorphic functions**, which are analytical functions with (well controlled) singularities. Two main results which express the transitions between the holomorphic and the meromorphic are particularly important in our perspective: **analytical continuation** and **Cauchy's residues theorem**. An analytical continuation allows to extend ("iterate") a given analytical function (on a well-behaved region of the plane **C**) to a meromorphic function (over a larger region, with singularities allowed). Cauchy's theorem allows expressing the value of an analytical function at a given *point*, through the calculation of values ("residues") of a small meromorphic variation of the function in the *boundary* of a region enclosing the point (for details and a wonderful *diagrammatical* presentation, see Needham (2004)).

As can be guessed from the preceding discussion, the *recto* and *verso* of Peirce's sheet of assertion may be modelled by the *analytical realm* and the *meromorphical realm* on the complex plane. This allows to model the *discontinuous Alpha cut* (which, in Peirce's original intuition, allows to pass from the *recto* to the *verso* of the sheet) as a (complex variable discontinuous) *operation* which allows to pass from analytical functions to meromorphic functions. Examples of such operations abound (the most natural being $f(1/z)$ or $1/f(z)$, which produce natural meromorphic functions associated to the analytical $f(z)$), and a specific choice of the operation depends on the additional Alpha rules that are required on the cut. Juxtaposition can also be modelled in different ways, for example through the product of functions, or through the product of its exponentials ($exp(f+g)$). Truth ("blank" in the *recto*) can be modelled by some sort of "smoothest" analytical function, for example $exp(z)$, and Falsity ("pseudograph" in the *verso*) by some sort of "wildest" meromorphic function, for example $exp(1/z)$[7].

[7] The singularities of the meromorphic functions are usually controlled by their negative degrees in power series expansions; if the degree is finite, the singularity is called a *pole*; if the degree is infinite, the singularity is called an *essential singularity*. The meromorphic function $exp(1/z)$ possesses an *essential singularity* at 0. An outstanding theorem in the theory of functions of complex variable (Picard's big theorem) asserts that $exp(1/z)$ attains all values (except at most one) in **C**, an *infinite* number of times, around *any* neighborhood of 0, however small. This *extreme* meromorphic behaviour can thus be very well related to Falsity. *Between* the extremes (Truth – Falsity), that is between $exp(z)$ and $exp(1/z)$, lies a profound hierarchy of

Alpha erasure (on a *recto* side inside a nest) corresponds to an identification of a product of analytical functions with an analytical one, while Alpha insertion (on a *verso* side) corresponds to the identification of a product of analytical and meromorphic with meromorphic. Finally, one can conjecture that the fundamental[8] Alpha iteration and deiteration rules originate a calculus of *homotopy classes* which is yet to be precisely explored.

The heuristics behind a complex variable interpretation of Beta depends on the understanding of the iteration of the line of identity *across Alpha cuts* as an analytic continuation of the line in the complex plane. The geometrical insight seems to be the correct one, since analytical *complex* continuation is *precisely* a specific procedure to pass from the analytical realm to the meromorphical realm, in perfect analogy with the procedure of extending the line of identity from regions with less cuts ("more" analytical) to regions with more cuts ("more" meromorphic). Here, the conjecture is that some (complex variable, topological) *calculi* related to analytical continuation may capture the (logical) deformations of the line of identity. A guide to a formalization of these *calculi* may be provided by Cauchy's residues theorem, *if* one can interpret a (logical) region with an Alpha cut as a (complex variable, topological) region with a *pole*. The lines of identity can then be modelled by affine bounded linear functions, and their crossings through Alpha cuts as meromorphic deformations of the line near the poles attached to the cuts. Then, on one hand, a calculus of singularities (residues) may provide a discrete rendering of the nested Alpha cuts, and, on the other hand, the Beta iteration of the lines of identity may be related to their analytical continuation around the boundaries of the regions with prescribed poles.

Going beyond classical first-order logic, Peirce's Gamma graphs help to diagram modal *calculi*. Peirce imagined two completely original ways to picture the modalities: using "tinctures" on the sheet of assertion, or constructing a "book of sheets" to enlarge our possible worlds. Since (an adequate) modal propositional logic is known to be equivalent to monadic first-order classical logic (modalities represented by monadic predicates), the *spreading* of regions (that is, extensions of predicates) in the complex

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analytical and meromorphic functions which should be used to model a logic of continuous truth-values beyond the classical discrete dichotomy. For a discussion of the philosophical issues that hinder our understanding of the passages between the continuous and the discrete, and for a study of the role of the existential graphs to facilitate that understanding, see Zalamea (2007).

[8] Caicedo has shown that iteration/deiteration is *the* fundamental adjointness that defines a general intuitionistic connective. See Caicedo & Cignoli (2001).

plane may then help to understand the *hierarchy* of modalities. Thus, beyond iterating the lines of identity (Beta), a spreading of regions (Gamma) encompasses some valuable underlying logical information, usually not considered. Here, in the theory of complex variables, many tools are available for the understanding of those "spreadings", and a connection between the logical aspects of the situation (modalities) and its geometric ones (representations, modulations, modularities) could provide astonishing new perspectives.

But perhaps the most promising path in the unravelling of a global complex variable interpretation of the graphs lies in interpreting Peirce's "book of sheets" as a full **Riemann surface**.[9] Beyond the *discrete* interpretation of the "book of sheets" as a Kripke model, on which the modalities receive their usual possible worlds semantics, the interpretation of the Book as a *continuous* Riemann surface provides many additional advantages. First, the Riemann surface *unifies in a single mathematical object* the various partial complex variable models for Alpha, Beta and Gamma. Second, a Riemann surface inherits a calculus of projections (see figure 1) which expresses, with new mathematical *content*, some logical correlations between propositional, first-order and modal levels. Third, a Riemann surface is a natural context for *blowing up*[10] in fibrations (see Petitot, 2003), which corresponds to constructing infinitesimal disks around blowed-up

[9] The concept of a Riemann surface (1851) answers in technical terms, in the complex variable situation, the general (philosophical) problem of glueing a Multiplicity into the One. In simple terms, the basic idea beyond a Riemann surface consists in representing a multivalent algebraic relation $r(z, w) = 0$ between a complex variable z (in the domain of the relation) and a complex variable w (in its codomain), thanks to a covering of the complex plane by a pile of planes, corrugated and holomorphically "glued", that represent the different possible values of w for given values of z. If, for a given $z0$, the equation $r(z_0, w) = 0$ has n roots, then n corrugated planes emerge ("sheets" of the Riemann surface) that cover the z-plane in a neighborhood of z_0. For some exceptional values of z ("ramification points"), the sheets are fused when the roots coincide, and the local expansions of w behave as fractionary powers of z (corresponding to the algebraic resolutions of the relation). Using the representations of the relations $w = z^{n/m}$ all usual Riemann surfaces (compact and oriented) can then be classified. Instead of working with the usual complex plane $\mathbf{C}$, it is easier to handle the projective plane $\mathbf{P}$ (adding a point at infinity), and, in that case, the Riemann surface of $w = z^{1/2}$ is homeomorphic to a sphere, while the Riemann surface of $w = z^{3/2}$ is homeomorphic to a torus. For a modern introduction (accessible and complete) to Riemann surfaces, see Fulton (1995), part X ("Riemann surfaces"), pp. 261-311.

[10] *Blowing-up* is a process to eliminate singularities in singular curves, introduced in projective algebraic geometry at the beginning of XXth century. If a curve in the complex plane possesses a singular point with ramification (different tangents at the point), the ramification comes to be *separated* in the blowing-up, and in the associated fibration the singular crossing is overcome.

points (an idea of Cartier), which in turn may correspond to infinitesimal spreadings of modal (that is, monadic first-order) properties.

Last but not least, the Riemann surface interpretation supports the understanding of logic that has emerged in the second part of XXth century. In fact, if a logic must be understood as a class of structures (Lindström) for which some logical axiomatizations serve as coordinatizations, and if the work of the contemporary logician lies in her tendency to look for fine invariants for classes of models (Shelah, Zilber), the unravelling of some mathematical *calculi* behind logical representations (as the complex variable *calculi* here suggested) explains also the *precedence of the geometrical under the logical*: a feature that Peirce constantly advocated, and that model theory and category theory are now proving with an extended range of new tools.

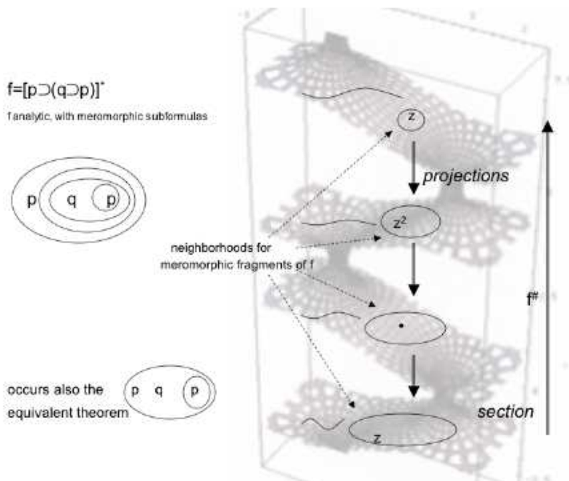

Figure 1: A Riemann surface for an existential graph

3. A category-theoretic perspective

The correspondence between the sheet of assertion with its logical transformations (codified in Alpha, Beta, Gamma) and the complex plane with its meromorphic transformations (along residues, analytic continuation and Riemann surface projections) is constructed over a deeper level of logical/topological correspondences, which becomes explicit when the logical viewpoint becomes **intuitionistic**. In fact, on the one hand, from a logical perspective, Peirce's graphs, which were constructed classically,[11] can be better understood intuitionistically (Oostra, 2011). And, on the other hand, from a topological perspective, the graphs capture some continuous combinatorics that cannot be reduced to discrete ones (Burch, 1991).[12] Thus, the *natural* surrounding where the graphs evolve is intuitionistic, both for logical *and* topological reasons.[13]

Brady & Trimble (2000a; 2000b) propose fine categorical models for *classical* Alpha and Beta, but nothing in their construction prevents to extend their ideas to the intuitionistic case (Zalamea, 2011). The boolean functors appearing in the proofs can be modified, and, instead of working in the category of Boolean algebras, their targets can be redirected to take values into three alternative categories: *(a)* the category of Heyting algebras (natural algebraic models for intuitionism), opening the way to intuitionistic fibrations; *(b)* the category of Stone spaces (natural topological models for intuitionism and modalities), opening the way to topological fibrations; or, *(c)* the category of subalgebras of meromorphic functions, opening the way to *complex variable fibrations*.

A functorial approach to these three alternatives, looking for their category-theoretic connections, is also related to the understanding of the intrinsic logic of sheafs of germs of holomorphic functions, a particular case of the **logic of sheaves** of first-order structures studied in Caicedo (1995). Caicedo has shown in fact that the logic of sheaves is intrinsically *intuitionistic*, and that it becomes classic only on very particular cases, depending on the structures at hand. In the case of the monoidal category of algebraic operators studied by Brady & Trimble, the sheafs definable over its natural *site* (in Grothendieck's sense) should turn out to be intuitionistic (Zalamea, 2011).

[11] Peirce's construction of the graphs is classical, mainly because classical logic was the logic emerging in Peirce's time (the influence of De Morgan on Peirce was determinant, for example, and De Morgan laws are a classical paradigm). Nevertheless, beyond the diachronic moment, Peirce's *natural* interest for a logic akin to topological considerations (continuity, synechism, neighborhoods, modalities) is permanent in his writings (see Havenel, 2006; Zalamea, 2001; 2003), and, in fact, well-behaved *intuitionistic diagrams* (without the name, but including the spirit) appear explicitly in Peirce's handwriting (see Oostra, 2011).

[12] Peirce's *theorematic* "reduction thesis" (as proved by Burch), with his *need* of the three categories, shows that the relational bonding of the graphs *cannot* be treated just as set-theoretic composition (where, using Kuratowski pairs, only two categorical levels are needed to reproduce the third).

[13] Since Tarski's Polish years, it is well known that the collection of opens sets in a topology is a sound and complete model for the intuitionistic propositional calculus. Lawvere showed many years later that the natural underlying logic of an elementary topos is also intuitionistic.

In that case, *a full circle of natural conceptual approaches to Peirce's graphs* would then be achieved, merging together the logical-topological-analytical-categorical, and inserting Peirce's graphs in the very core of mathematical knowledge (sheaf theory and complex analysis).[14]

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