1 "The form of a relation": Peirce and Mathematical Structuralism

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In this paper, I shall argue, first, that, in his philosophical writings on mathematics, Peirce confronted problems that are similar to those that exercise contemporary philosophers of mathematics, and, second, that his responses to these problems, in particular his claim that the object of mathematical reasoning is "the form of a relation," show that he accepts a version of a position that is now referred to as mathematical "structuralism" (Resnik 1997; Shapiro 1997).

Section one introduces some problems about our cognitive access to mathematical objects, and section two describes how those problems arise in Peirce's philosophy. In section three we introduce Peirce's claims about "the form of a relation" and then, through considering some of his claims about natural and rational numbers, make the case for his being a mathematical structuralist (section four). This is followed by a discussion of how he should understand structures (section five) and an examination of his views about how we refer to objects such as numbers. The final section draws some conclusions about our access to mathematical reality.

1. Some Metaphysical Questions about Mathematics

Much contemporary work in the philosophy of mathematics has been preoccupied with a range of issues about the nature of mathematical *objects*, things like numbers and sets. We use expressions that appear to refer to such objects: 'seven', 'the square root of nine', 'the set containing just the books that are now on my desk' and so on. And mathematics appears to study the properties of such objects. But they differ from more familiar material objects in several important ways, which are all related to the fact that such objects are abstract rather than concrete. First, they are not things of which we can have sensory experience; second, they are causally inert; and third, they appear to lack any location in space and time. In Peirce's terminology, they lack "secondness," the capacity to interact causally with other things and with people. Yet they also appear to be objective and mind-independent: common-sense would tell us that *two plus three equals five* was true long before anyone existed who was in a position to understand that proposition or to recognize that it was true. And since this objective truth appears to be about objects such as *two*, *three*, and *five*, it apparently follows that these objects must be mind-independent denizens of reality, and, what is more, they are things about which we can obtain knowledge.

These assumptions about the subject matter of mathematics can seem problematic for at least three reasons. The first problem is straightforwardly metaphysical: abstract objects differ from familiar kinds of everyday objects, and it can seem problematic how there can be such things. The point may be put in terms of truth makers. What *makes it true* that two plus two is four? Abstract objects seem ill fitted for this kind of work and they just seem mysterious. A second issue concern reference: how is it possible us to talk or think about abstract objects? When we refer to ordinary physical objects, we exploit the fact that we can act upon the objects and they can act upon us. I can refer to the tree in my garden because I can point at it, exploiting the fact that it has a spatial location relative to me. I can refer to invisible concrete objects, but I do so by exploiting their causal interactions with objects that I can perceive; I know them through their effects. In each case, my ability to think about the objects exploits features that are absent from numbers, sets and other abstract objects: my ability to think about concrete things depends upon the fact that they possess what Peirce called secondness.[2] The third issue concerns how we obtain knowledge of such things: I obtain knowledge of the properties of the tree by looking at it, by carrying out experiments upon it, by moving closer to it. My knowledge of the tree (or any concrete object) relies upon properties that it does not share with abstract objects. The problem of how we have knowledge about (and perhaps refer to) abstract objects is sometimes referred to as the "access problem" (see (Macbride 2007)). This problem is widely thought to be insuperable and many philosophers accept the view, often called "nominalism," that there are no such things as numbers, sets, and other abstract objects.[3]

Responses to these kinds of questions in contemporary philosophy of mathematics employ several strategies. Some philosophers, sympathetic to the belief that there are no real abstract objects, but also recognizing that our most familiar ways of expressing mathematical propositions either involve quantification over abstract objects or involve explicit reference to abstract objects such as sets and numbers, have become sympathetic to *fictionalism*. Since there are no numbers, arithmetic is *false*; since there are no sets, set theory is *false*. Statements about them are all fictions, but they may have valuable applications (see, for example, (Field 1980)). A second strategy is to claim that the surface structures of sentences about mathematical matters are often misleading. We can paraphrase mathematical propositions in ways that involve no reference to or quantification over, abstract objects. A third strategy is to embrace Platonism: there is nothing problematic about the idea that there are real, knowable abstract objects. This strategy incurs the obligation to explain just how we can talk about, and obtain knowledge of, such things.

2. Objects and Interpretants

The various forms of the access problem arise when we notice that our access to external empirical objects depends upon their possessing features which are, apparently, lacking in the objects of mathematics such as sets and numbers. This makes it problematic how we can refer to, and obtain knowledge of, these mathematical objects. In this section we shall identify two ways in which such problems arise in Peirce's philosophy.

Peirce treated the fundamental semeiotic or semantic concepts as irreducibly triadic. Signs have objects, but the relation between a sign and its object is mediated through a process of interpretation or understanding. The name 'Paris' denotes that French city because it should be interpreted as a sign of it. In "Prolegomena to an Apology for Pragmaticism" he gives an analysis of "the essence of a sign" as "anything which, being determined by an object, determines an interpretation to determination, through it, by the same object" (Peirce 1906b:C4, pp. 413-414). Suppose that the clouds are a sign of rain, then observing them will lead me to thoughts about the rain—for example that I should take an umbrella. And if someone tells me something using the name 'Paris', then my interpretation of what they say will give me information about Paris itself. Issues about numbers then might be raised as questions about what the objects of mathematical expressions are or as questions about how we can interpret mathematical expressions. I understand '3+4=7' as a sign of a mathematical state of affairs, and I understand the numeral '3' as a sign of the number three. Using a non-Peircean vocabulary, an explanation of what expressions such as numerals *refer* to will also provide an explanation of how we can *understand* or *interpret* them.

We begin with an example of knowledge of concrete physical objects, presenting, first, an issue about reference and, then, a problem about knowledge and interpretation. Peirce's account of reference distinguishes two kinds of objects, "immediate" and "dynamical":[4] most linguistic expressions, especially singular terms such as 'Paris', will have objects of both kinds. When I refer to a concrete object, the words or other signs that I use will present that object in a distinct way: I may refer to something as "that blue book" or as "the sheep on that hillside" or as "the morning star." These formulations identify the immediate object of my utterance, specifying the object *as it is presented in the sign*. The dynamical object is the thing itself that I refer to, which may turn out not to fit the characterisation contained in the immediate object. In a draft of a letter to William James, Peirce describes it as "the Object in such relations as unlimited and final study would show it to be."[5] What I identify as a sheep on the hillside in murky weather may turn out to be a bush or rock. For some speakers, 'The morning star', 'the evening star' and 'Venus' may have different immediate objects but the same dynamical object. There is room for a gap between things as I represent them as being and those things as they really are: the immediate object may actually misrepresent the real or dynamical object of the sign. This raises questions about what "anchors" a sign to its real or dynamical object when the sign actually misrepresents that object. In most cases, especially when concerned with concrete objects, the dynamical object is likely to be something I have entered into causal interactions with. This is especially clear when I refer to something using a demonstrative expression: I can judge that *That is a sheep* while pointing to what is, in fact, a tree. In our dealings with concrete objects, we can correct our immediate objects because we enter into real physical, causal or sensory relations with the things we are thinking about, and these causal relations between signs and their objects constrain our interpretations of them. Is this a reason for denying that we can really refer to mind-independent mathematical objects? Abstract objects lack secondness: we do not enter into causal interactions with them. (See (Hookway 2000:Chapter 4).) Hence the sort of anchorage that guides us in identifying the dynamical objects of signs that refer to concrete objects is not available when we are talking about mathematical objects. Abstract objects are problematic because they are causally inert; indexical reference is our primary kind of reference to concrete objects but, it seems, indexical reference to sets and numbers is problematic.

Peirce's philosophy enables him to distinguish abstract objects from concrete ones. He thinks that both are real but that only concrete objects exist. Existents "dynamically react against other things" (Peirce 1896b:C1, p. 248), entering into causal relations, possessing the characteristics which Peirce calls secondness, thus sowing the suspicion that where there is no "dynamical reaction" there is no dynamical object. Numbers and sets lack secondness, but this does not mean that they cannot be real. It is sufficient for numbers to be real that there are true sentences that contain numerals: if it is true that 7 is a prime number, then there really are numbers. This provides one thread in Peirce's response to the problem that I have described. But there is more work to be done. Peirce's pragmatist account of truth holds of true propositions that anyone who inquires into them, long enough and well enough, is "destined" or "fated" to end up accepting them. Our acceptance of such propositions is constrained, as is our recognition that our current conception of some object may misrepresent its true character. We can still use the distinction between immediate objects and dynamical or real objects. If mathematical objects are real yet lack existence, we still need an explanation of how we can be constrained in forming our opinions about them and revising our conception of their properties. How is our thought about, for example, numbers anchored if mathematical objects lack secondness?

Another version of the access problem can be raised using Peirce's maxim of pragmatism, which is a rule for obtaining reflective clarity about the contents of propositions or concepts. We obtain such clarity by identifying what "practical difference" it would make if the proposition was true or if the concept applied to something. In some formulations, Peirce insists that a concept or proposition can make such a difference only if its application or truth would make a difference to what sorts of sensible experiences we should have. Such clarification is important because it provides information relevant to how we can test hypotheses. Peirce sometimes uses his maxim to expose the errors of "ontological metaphysics": if the truth of metaphysical sentences makes no difference to experience, then the metaphysical theories are empty. One route to scepticism about mathematical objects involves arguing that since the truth of a mathematical sentence is not observable, then mathematics is empty, making no respectable claims about reality. This concern is, in some respects, similar to anxieties about mathematical knowledge that depend upon the logical positivists' verification principle. If we don't know how to verify mathematical sentences empirically, then it makes no sense to think that we understand them, and, furthermore, it seems that we cannot have knowledge of them.

We have identified two grounds for concern about mathematical knowledge and about the reality of mathematical objects that are based on some important themes in Peirce's accounts of understanding and reference. We now turn to his positive claims about the objects of mathematical knowledge.

3. Molecular Structures and "The Form of a Relation"

In "Prolegomena to an Apology for Pragmaticism," a 1906 draft for a paper intended to contribute to a proof of the correctness of the pragmatist maxim, Peirce tried to show that what goes on in mathematical reasoning is much more like what goes on in ordinary scientific reasoning than might have been supposed. The relation between a chemist and her subject matter bears strong analogies to the relation between a mathematician and her subject matter. If this is correct, then perhaps the sceptical arguments we presented can be answered. The crucial step of the argument involves two claims: first, that mathematical reasoning is diagrammatic reasoning; and, second, that when we engage in such reasoning, the "Object of Investigation" is "the *form of a relation*" (Peirce 1906b:C4, pp. 411-412). But before explaining what this means, we must describe what goes on in research in chemistry.

When the chemist carries out experiments, she is "putting questions to nature," and, as we might say, trials are "made upon the very substance whose behaviour is in question." But it is important that what we are investigating is not just a particular sample. Rather the chemist is studying "the molecular structure" and the chemist was "long ago in possession of overwhelming proof that all samples of the molecular structure react chemically in exactly the same way; so that one sample is all one with another" (Peirce 1906b:C4, p. 412). The object of the experimental research is this Molecular Structure; and since it is present in every sample, the researcher experiments upon the "Very Object under investigation" every time that she experiments upon a particular sample. We can use the particular sample because it can serve as an iconic representation of any other sample, and this is because they share a feature, namely the molecular structure. Experimental research works—and avoids the sceptical arguments described in the previous section—because (1) we can interfere with the chemical sample by subjecting it to experiments, (2) the structure makes a causal contribution to the effects that our interventions have, and (3) our interventions have consequences that are observable.

Rather than turn directly to the case of mathematical reasoning, we should first consider a more concrete example of what Peirce calls "diagrammatic reasoning." A map provides a diagram of some particular terrain and, in familiar ways, we can use it to gain information about that terrain. According to Peirce, this, too, can involve experimentation. A very simple example: by measuring the distance on the map between the dots representing two towns (A and B) and comparing this with the result of measuring the distance between one of those dots (A) and a third one (C), we can discover whether A is closer to B or to C. It is easy to construct far more complicated examples too.

In this case, it is natural to think that I do not experiment upon "the very object of investigation." The object of investigation is the terrain, and my experiments are directed at a distinct object (the map) that merely represents the terrain. I learn something about the terrain, and, relying on some similarities between the map and the terrain, I draw conclusions about the latter from what I have learned about the former. There is surely all the difference in the world between experimenting upon "the very object under investigation" and experimenting upon some surrogate for it; and when we use diagrams as tools for our inquiries, this seems to involve using a surrogate intermediary between the investigator and the object of her inquiries. In that case, if mathematics is, indeed, a discipline that involves diagrammatic reasoning, then there is a fundamental difference between the activities of the chemist and the activities of the mathematician. The gap between the diagram and object of investigation may be just the sort of thing we need to argue for scepticism about the objects of mathematics.

The main conclusion of Peirce's discussion is that *this* way of looking at things is a mistake. In both cases, we experiment upon "the very object under investigation." The only difference we have to contend with is between the kind of feature is that is present in all the different samples the chemist investigates and the kind of feature that is present in both the diagram and the things that the diagram represents: in one case it is a shared molecular structure; and in the other it is a shared relational form. Just as different samples of a substance share a molecular structure, so the map and the terrain that it portrays share a structure of a different kind. The terrain, we might suppose, contains towns that stand to each other in relations of distance and direction; and the map may contain dots that are related to each other in terms of relative position on the map. But each is an instance of a more abstract pattern of relationships that ensures that we can make further discoveries about the terrain by exploring corresponding relationships on the map (and we can also make predictions about the properties of the map by making observations of the terrain).

Recognizing these parallels between the molecular structures of the substances and the abstract relational structures that underlie isomorphism between diagrams and the things they represent should not prevent our taking seriously the differences between them. Maps depend upon conventions, for example, while nothing like that is involved when a substance has a particular molecular structure. It is also important that the existence of these abstract isomorphisms between the marks on a piece of paper and the characteristics of the terrain is independent of our being prepared to use the paper as a map or recognize the possibility of doing so. The shared structure can, in principle, be studied in abstraction from the possible applications of the resulting knowledge. If the similarity between chemistry and diagrammatic reasoning is to be established, then we have to hold that the object of our investigation when we engage in diagrammatic reasoning is neither the map nor the terrain but rather that *shared structure* that is embodied in each. We study the abstract structure, and, since that structure is present in both the map and the terrain, we also study the map and the terrain. Moreover, since we are studying an *abstract* structure, our reasoning when dealing with maps is already mathematical. What we learn is applicable to any system that embodies the structure in question, whatever intentions we might have for the using the knowledge we obtain.

The concrete relations that obtain between places in the terrain are different from those that obtain between locations on the map; and the relations between the dots on different maps may differ too. But it is compatible with this that there are abstract similarities between these sets of relation which allow us to use each as an iconic sign of the other. In doing this we understand the terrain and each of the maps that we use as a sign of the abstract formal relational structure which is realised in different ways in each of its "instances." We can study the shared structure, abstracting from the individual maps or from the terrain and when we do that our inquiry is more explicitly mathematical. But even then, we can usually study the abstract structure only by examining its concrete instances. Since these concrete instances are likely to be observable, and since we are able to manipulate them and experiment upon them, we are no more remote from this common formal structure than we are from the molecular one. If the analogy that Peirce draws here is sound, then it may be possible to see how we can have access to mathematical objects: we study forms of relation by examining concrete objects that exemplify them. How numbers and sets are related to such forms would need further explanation, and this is a topic that we shall consider further below.

4. "The Form of a Relation"

There is a problem lurking here. Peirce's talk of "the *form* of a relation" places a restriction on the kinds of shared features that provide suitable objects for *mathematical* reasoning: the different systems that are instances of a given mathematical structure have a common *form*. What does this mean? What kind of *form* provides objects for mathematical knowledge?

A contemporary philosopher of mathematics who is sympathetic to mathematical structuralism is Stewart Shapiro (Shapiro 1997:5-6): "The subject matter of arithmetic is the *natural number structure*—the property common to any system of objects that has a distinguished initial object and a successor relation that satisfies the induction principle." Indeed, "the essence of a natural number is the relations it has with other natural numbers." It is natural to describe this view as one that holds that mathematics is concerned with "forms of relation."

As Shapiro admits (Shapiro 1997:98-99), clarifying this notion of *form* is extremely difficult. We may not able to do this without distinguishing logical features of a system of relations from its non-logical features, the formal features being the logical ones. And it may be difficult to see how we can identify the *logical* without already making use of a notion of *form*, in which case we are in danger of begging the question. One feature that contributes to making a structure "formal" or "mathematical" is that it is "freestanding" (Shapiro 1997:100): whether an object fills a particular place or "office" in the structure "is characterized entirely in terms of how it relates to other offices of the structure." There are no external conditions upon being eligible to fill one of these places. Thus *any* object can fill the "office" of being the number six in some system; but you can only fill the office of being American President if you are an adult American citizen. The same might hold of systems which instantiate the formal structure of a map of Sheffield or the Sheffield terrain. Systems which use the Sun to fill the same role in the structure as some particular dot on my map would not be very convenient for use *as maps*, but that does not mean that they do not instantiate the formal structure that the map displays. In order for something to indicate a town on a map, something may need to have non-formal properties: it must be the kind of thing that we can observe, for example. But while these non-formal properties are necessary for belonging to an instantiation of the structure that can be used as a map, they are not required for simply being part of an instance of *that structure*. There are indications that this is how Peirce understands the *formal* (see (Murphey 1961:235f). It is suggested by remarks about "the dignified meaninglessness of pure algebra" and by the claim that:

> In pure algebra, the symbols have no other meaning than that which the formulae impose upon them. In other words, they signify any relations that follow the same laws. Anything more definite detracts needlessly and injuriously from the generality and utility of the algebra. (Peirce 1902e:C4, p. 256)

But while it is suggestive, Peirce does still not make the desired notion of the formal very clear.

Illuminating as the example of reasoning with maps is, it is not a typical example of what we would think of as *mathematical* reasoning. Are there examples of the same phenomenon in areas of inquiry that are more typical of mathematics? There are a number of passages in which Peirce tries to clarify what he refers to both as "whole" numbers and as "cardinal" numbers by giving an account of a distinctive kind of series and arguing that "the only thing that whole numbers can express is the relative place of objects in a simple, discrete, linear series" (Peirce 1905d:C4, p. 275). One simple formulation is:

> First, there is a relation, *G*, such that to every *number*, i.e. to every object in the series, a different number is *G* and is *G* to that number alone; and we may say that that to which another number is *G* is "*G*'d" by that number. [I.e., every number has one, and only one, successor.]
>
> Second, There is a number, called zero, 0, which is *G* to no cardinal number.
>
> Third, The system contains no object that it is not necessitated to contain by the first two precepts. That is to say, a given description of number only exists provided the

> first two precepts require the existence of a number which may be of that description. (Peirce 1897d:C4, pp. 136-137)[6]

This formulation, along with others that Peirce uses, is similar to the Dedekind-Peano axioms.[7] As it stands it is not exactly the same, not least because, like Frege, Peirce appears to have supposed that he could prove the principle of mathematical induction from his initial characterization of the series, thus avoiding the need to include the principle of induction as an additional fundamental postulate. In this, he is not alone. Frege, by using second order logic to define the ancestral, was able to define the natural numbers as, in effect, "those things for which mathematical induction holds" (Frege 1884:Sections 79 and 83) (cf. (Dummett 1991:12, 120-121). We saw above that Shapiro's structuralism defines the "natural number structure" as "the pattern common to any system of objects that has a distinguished initial object and a successor relation that satisfies the induction principle" (Shapiro 1997:5-6). In Peirce's writings from around 1880, something like the principle of induction was offered as an independent axiom (Peirce 1880-1881). In later work, he undertook to prove the principle of induction from the other axioms.[8]

The definition ensures that that we cannot derive from these "precepts" any properties of whole numbers that involve any concept that is not involved in the concept of such a series. There are many concrete simple discrete linear series whose members have their own distinctive characteristics; but these are irrelevant to their suitability for the role of whole numbers. One qualification is required here. Since there are infinitely many natural numbers, it is highly controversial whether there are any concrete series that exemplify in its entirety the natural numbers structure. What we can be sure of is that there can be concrete structures that provide *partial realizations* of the structure. This need not undermine the substantial claims being made here. The current conclusion is that all is required for the arithmetic of whole numbers is a series with a formal character; and that formal character is manifested, perhaps only partially, in all of the concrete series. This appears to ensure that the whole numbers are freestanding.

Peirce holds that his sort of approach is of quite general application. In a manuscript entitled "Another Curiosity" (Peirce 1908a), he provided a detailed discussion of the rational numbers. I shall not examine or evaluate the details of this, but it is useful to notice his general characterization of what he is trying to do. The argument exploits a distinction between the fractions and what Peirce calls the "rational values" and we would be more likely to refer to as the rational numbers. He thinks of the fractions as "expressions" that denote the rational numbers. And we need the distinction because there can be different fractions that denote the same rational number: for example, $\frac{1}{2}$ and $\frac{5}{10}$ are distinct fractions that denote the same rational.

Although Peirce expresses this view by treating fractions as linguistic expressions, this may not be strictly required by his argument: the important thing is that different fractions can pick out the same rational numbers. Fractions are the same if, only if, they have the same numerator and the same denominator.

In his paper, Peirce defines, first, the fractions, and, second, the rational numbers. On each occasion, he does so by giving a rule for the construction of the appropriate series. He tells us that "the entire doctrine of fractions is contained in this series or rather in its governing definition or rule of construction" (Peirce 1908a:C4, p. 578). And later, he tells us that "All the properties of rational [numbers] and of their expressions in their lowest terms follow from the general fact that they are all contained in their order in the series constructed according [to the rule he stipulates]" (Peirce 1908a:C4, p. 580). He thus explicitly recognizes that the fractions and the rational numbers are freestanding. And he insists that all that is involved in the rational numbers is the form of "relation of rational consequence" that he has described.

5. Peirce and the Varieties of Structuralism

The literature of contemporary structuralism has yielded a variety of different versions of the position, differing mostly in terms of how propositions about structures are to be understood. A brief survey of these various versions of structuralism will help us to identify some important features of Peirce's view, although, of course, we should bear in mind that he was writing many years before some of these distinctions entered the literature. It also contains criticisms of the structuralist position, often questioning whether structuralism is any more successful than more extreme Platonist positions in accounting for our ability to refer to, and acquire knowledge about, mathematical objects.

We need to distinguish structures from *systems*. A system is just a set of objects standing in certain relations (Shapiro 1997:73-74). An *in re* structuralist holds that "there is no more to structures than the systems that exemplify them"; while an *ante rem* structuralist is a realist about structures who holds that the reality of structures does not depend upon there being any systems that exemplify them.[9] The former option is likely to be attractive to nominalists who are reluctant to accept an ontology of abstract objects. As Shapiro puts it, it offers a position that can be described as "structuralism without structures" (Shapiro 1997:85). It is a major problem for the *in re* version that, in order to deal with structures of real numbers, natural numbers, or the set theoretic hierarchy, it requires a very large ontology of objects that can form parts of these systems. The major difficulty for the *ante rem* position is that it needs to explain what the reality of the required structures consists in when there are no systems that exemplify them.

Charles Parsons has distinguished two sorts of *ante rem* structuralism. Our ordinary talk of structures suggests that they are objects of some kind. We talk of "the natural number structure" and issues are raised about whether structures "exist" (Parsons 2004:59). The alternative, favoured by Parsons, is to derive our conception of a structure on the basis of what we do when we "specify a structure": this involves "giving a predicate for the domain and additional predicates and functors, together with certain conditions involving them" (Parsons 2004:59). This suggests that the structure is something general rather than a particular. In specifying the structure we provide a rule for determining the shared properties of any system that exemplifies that structure.[10] In that case the structure is something general rather than being (like systems) a complex object. And when we appear to talk about structures as objects, this has a secondary meta-linguistic character. So let us distinguish structures as objects from structures as complex properties.

Phrases like "the form of a relation" suggest that Peirce is concerned with some sort of second order property rather than with an individual when he identifies the objects of our mathematical investigations.[11] This is reflected in his philosophical struggles with 'nominalism'. Traditionally, the word was used to identify a response to the problem of universals, of what Peirce would call "*generals*." It holds that the only things that are real are individuals, but it is compatible with this that some of those individuals are abstract. In contemporary metaphysics it is often used for the denial that there are any *abstract* individuals, such as sets or numbers. Peirce is strongly realist about generals, and rejects the nominalist thesis the only things that are real are particulars: there is real thirdness. Thus the idea that structures are general (rather than particulars) would be congenial to his realism.

Other passages support this interpretation. Passages suggesting that Peirce would favour *ante rem* structuralism include: "All the properties of rational [numbers] … follow from the general fact that they are all contained in their order in the series constructed according to the rule" that Peirce formulates, and he insists that all that is involved in the rational numbers is "the form of relation of rational consequence" that he has described (Peirce 1908a:C4, p. 580). The properties of the numbers (in this case the rational numbers) are determined, not by the membership of a particular system, but rather by the rule according to which the "series" or system is constructed. A rule is a general, and Peirce is robustly realist about generals; and the passage suggests that the properties of the members of the series are determined by something that is real, independently of the existence of any system that is determined by it.

There may also be passages that suggest that he was an *in re* structuralist. Those who want to treat mathematics as dealing with systems rather than abstract objects will often treat mathematical propositions as having a hypothetical character. Suppose that Φ is a sentence in the language of arithmetic; then its content can be captured as follows:

For any system *S*, if *S* exemplifies the natural number structure, then $\Phi[S]$

Where $\Phi[S]$ is obtained from Φ by interpreting the non-logical terminology and restricting the variables to the objects in *S* (taken from (Shapiro 1997:86)). We understand the arithmetic claim as making a hypothetical claim about systems rather than as making a claim about an abstract structure. It is significant that Peirce often says that all mathematical propositions are hypothetical. For example, he tells us that "Mathematics is the study of what is true of hypothetical states of things" (Peirce 1902e:C4, p. 193); and in (Peirce 1896d:C1, p. 23): "[M]athematics is only busied about purely hypothetical questions."

It is not easy to sort out just where Peirce stands on these issues.[12] I suspect that what is available *ante rem* is a rule which determines what would happen in all possible cases. Even if there are no existing systems that exemplify a given structure, there can be a rule determining what properties such systems would have if there were any. Although Peirce sometimes suggests that mathematical propositions are about systems (actual and possible), their truth-values are determined by this rule. What makes it the case that different systems exemplify the same structure, in that case, is not merely the fact that they are analogous, but the fact that are constructed according to the same rule. Moreover we see here an answer to our question about the dynamical objects of mathematical propositions: the structuralist view gives an answer to the question what numbers (for example) are. Moreover the properties of such objects are determined by the rule that determines the constructions of the relevant structure or series. This leaves open the possibility that we may be mistaken about the properties of these objects, thus making room for a distinction between the immediate and dynamical objects of mathematical expressions.[13]

Structuralism does not lack critics (see (Hale 1996; Macbride 2007)), and the following two sections will discuss how Peirce responds to two of these. First, even if we accept the account of mathematics suggested by Peirce's writings and agree that numbers, for example, are places in a series, how can we make sense of our ability to refer to numbers using numerals or other singular terms? How can we talk about numbers? Second, the account of diagrammatic knowledge introduced in section three suggests that arithmetical knowledge, for example, depends upon our ability to experiment with *concrete* instances of the appropriate formal structure.

Does that mean that arithmetical knowledge would be possible only if the world contained a denumerable infinity of concrete objects? This would be very implausible.

6. Talking about the Natural Numbers

According to Peirce, the "cardinal numbers, strictly understood, are vocables or written signs, of which one is attached to each finite multitude" (Peirce 1908a:C4, p. 556). This claim is suggestive, but it is also curious for two different reasons. First, Peirce does not seem to be sufficiently sensitive to the distinction between numbers and numerals. It seems more accurate to distinguish the numbers from the "vocables" and written signs that are used to express or denote them. Peirce's unease with this is reflected in the fact that he sometimes writes of "cardinal numbers" and at other times of "cardinal numerals." Treating numbers as written signs has the unfortunate consequence that 'two', 'dos', and 'deux' count as different numbers. These different words are surely different terms for the *same number*. But, second, it may be suggestive through its unclear recognition that numbers aren't the kind of objects that have non-mathematical attributes as well as mathematical ones; they are numerical expressions to which there is nothing apart from their use as devices for counting.

Given his account of how these numerical expressions enter the language, this is the more plausible view. An explanation for all this is suggested when Peirce offers an explanation of how just how terms for numbers entered our language. The origin of our arithmetic talk lies in the vocables that we use for counting. He suggests that the differences between these numerals and the sounds used in children's counting games such as "Eeny, meeny, miney mo" are not great.

The only essential difference is that the children count on to the end of series of vocables round and round the ring of objects counted; while the process of counting a collection is brought to an end conclusively by the exhaustion of the collection, to which thereafter the last numeral word used is applied as an adjective. This adjective thus expresses nothing more than the relation of the collection to a series of vocables. (Peirce 1897d:C4, pp. 133-134)

So we introduce 'one', 'two', 'three' and the rest into the language, not as expressions that express arithmetic conceptions, but rather as sounds which have a place in a counting game. Like the children playing the "eeeny, meeny" game, we play counting games using what we now recognize as numerals. The numerals are not expressions that refer to the numbers that we study in arithmetic: all the game needs is the meaningless vocables. If we watch French and Spanish children counting, we notice that they use 'deux' and 'dos' in the game at the same step that we would use 'two'. They have the same role in counting, but it would be misleading to say that they express the same concept, and perverse to describe them as "synonymous."

The second curious feature of the passage is that it identifies the counting vocables with *cardinal* numbers, rather than with the ordinals. Since the operation of counting imposes an ordering on the set of things which are being counted, it is natural to treat the concept of an ordinal number as more fundamental than the concept of a cardinal number. Dummett criticizes Frege for failing to see this (Dummett 1991:293). Elsewhere Peirce avoided this error, emphasizing that children's counting "vocables" are "ordinal" (Peirce 1908a:C4, p. 559) and endorsing Dedekind's claim that "the doctrine of ordinal numbers" should be "made to precede the cardinals" (Peirce 1901a:C3, p. 400).[15]

Peirce informs us that when, on counting the members of a collection, we terminate at 'seventeen', for example, we have the habit of transforming the counting vocable into an adjective. Instead of saying that the collection was such that the last counting vocable we use in counting its members is 'seventeen', we say that there are seventeen objects in the collection. And having done this, he continues, "there is a real fact of great importance about the collection which is at once deducible from that relation." This is that:

> The collection cannot be in a one-to-one correspondence with any collection to which is applicable an adjective derived from a subsequent vocable, but only to a part of it. (Peirce 1897d:C4, p. 134)

The fundamental thought here is that such facts as these are all we require in order to treat "quantities" as attributes of the collections themselves. It provides an explanation of how "meaningless" noises used for counting can give rise to adjectives that can be used to say something substantive about the collection. Counting gives way to adjectives; and once we have adjectives, it becomes possible to draw inferences from the fact that that adjective is applicable to something.

In his paper, Peirce seems to suggest that "the cardinal numerals" are "meaningless" but in a respectable or useful sort of way. The only use that the cardinal numbers have, he suggests, is "to count with them and … to state the results of such counts" (Peirce 1897d:C4, p. 136). The move to an adjectival use of numerals from which "great facts" that are potentially surprising can be deduced requires a substantial shift in the ways in which we use the numbers. As Peirce acknowledges (Peirce 1897d:C4, p. 134), we cannot actually derive the great fact from the simple adjectival use that was originally described. The simple counting use of numerals together with the simple adjectival use was "developed during the formative period of language." Before we can begin to derive "great facts" from our use of number-expressions, it is necessary that the simple number system be "taken up by the mathematician, who, generalizing upon them, creates for himself an ideal system" (Peirce 1897d:C4, p. 136).

So what goes on at this stage? There are two important developments. To understand the first, we need to recall that the series of numerals (counting vocables) itself forms a simple, discrete, linear series, each numeral having its place in that series. Indeed, we can recognize that there are rules that can be used to generate this series of numerals. The mathematician recognizes that there can be many other series with the same form and tries to explain what that form is and to investigate its properties. At this point, we can use the counting vocables as a diagram of any other series with that form, and we can use any of those other series as diagrams of the vocables, and so on. Each such series is an iconic representation of the others. The mathematician's diagrams can then represent that form itself, of which each series is an instance. The fact that the counting vocables are themselves an instance of the "form of relation" unifying the whole numbers may contribute to Peirce's tendency to slide between talking of numbers and talking of numerals.

This process involves an inferential pattern that Peirce refers to as "hypostatic abstraction," one that he takes to be essential to mathematical advances. (See (Hookway 1985:201ff), and (Short 2007:264ff).) It enables us to begin with an expression that is used "adjectivally" and derive from it a "noun substantive," a term which treats what was previously expressed adjectivally as a substance or object. One of Peirce's examples involves the inference from 'Honey is sweet' to the conclusion that 'Honey possesses sweetness' (Peirce 1902e:C4, pp. 194-195). A similar shift occurs when we treat series and forms of relation as *objects of study.* We can see what this involves using the example of the natural numbers. Compare the following stages in our dealings with this series.

1. First, we simply use counting vocables each of which has a place within an instance of the series in question.
2. Second, we introduce simple numerical adjectives, describing classes by reference to the last vocable we have to use in counting their members.
3. Third, the mathematician abstracts from this practice and recognizes the series, and its formal structure, as an object of study.
4. Fourth, when describing this practice, we transform these numerical adjectives into names that have numbers as their objects. Peirce wrote that "positive whole numbers can express nothing but places in a linear series" (Peirce 1905d:C4, p. 275). Although 'express' is vague here, it is natural to

interpret this as claiming that a place in the linear series is simply *what the number is*; such places are what numerals refer to.

5. Talking in this way greatly increases our opportunities for making generalisations about the series and also enables us to see such series as instances of yet more abstract structures.

The parallels between what Peirce says and the views of a structuralist such as Shapiro should now be manifest, especially in the light of the remark that numbers simply mark a place in the series.[14] He is also able to bypass the sorts of metaphysical worries that we noted in the first section: once abstraction has done its work, there will be true sentences containing numerals and other "noun substantives." And, for the pragmatist, this is enough to make it the case that numbers are real: "On pragmatist principles reality can mean nothing except the truth of statements in which the real thing is asserted" (Peirce 1903n:N4.161-162). But this does mean that numbers lack *existence*, they cannot interact with anything. Indeed, this is reflected in the kind of mode of being that abstractions have: the being of a number, for example, "merely consists in the truth of some proposition concerning a more primary substance." (Peirce 1903n:N4.162).[16]

We can now see where Peirce should stand on the dispute that was mentioned in the previous sections, between *ante rem* structuralists and their *in re* opponents. When describing a *system*, the forms of relations that hold the system together are characterized in an adjectival manner; we refer to the objects and describe their relationships. But a step of hypostatic abstraction will enable us to advance from this adjectival attribution of the relational structure to reference to a substantive characterization of "the structure." And the mode of being of the structure consists in the truth of some propositions concerning more primary substances, namely truths about some systems which are instances of that structure.

One implication of this, at least in the case of the numbers, concerns the objects of mathematical signs. The immediate object of a numeral is a place in the number series. Since the series of numerals is an instance of the structure of the numbers, and we cannot understand one numeral without grasping its place in the continuing series of numerals, the numeral presents the number to us as occupying a particular place in the series. Although we may learn more about that number's properties (for example that it is odd or prime), this does not provide for the possibility that the *dynamical* object of my use of a numeral is any different from the immediate object. Numbers do not have a mode of being which is not constituted entirely by their mathematical properties. And, such is the simplicity and familiarity of the natural and rational numbers, it is easy to suppose that anyone who can use such numbers has an implicit grasp of the relevant structure and the place of the numbers in it. There is thus little room for a gap to open between the immediate and dynamical objects of ordinary numerals.

In other cases, the immediate objects of mathematical expressions may fail to reflect constitutive features of their dynamical or real objects. For example, someone can acquire the grasp of a complex number by being introduced to simple examples such as the imaginary number *i*, the square root of minus one. They could do this, and even use *i* in calculations, without having any grasp of the structure to which it belongs and of the place in that structure which determines its identity. Once the mathematician gets to work, we learn that the complex numbers (including the imaginary numbers) form a field, each number being represented as an ordered pair of real numbers, and are subject to some distinctive operations:

$$(a,b) + (c,d) = (a + c, b + d)$$
$$(a,b) \cdot (c,d) = (ac - bd, bc + ad)$$

Many of us do not understand just what sort of structure the complex and imaginary numbers form. But it will still be the case that an expression such as i marks a place in a structure and the numbers are freestanding, having no attributes which are not mathematical. The example of complex numbers provides a nice illustration of how we can use concepts in the absence of a clear grasp of what their dynamical objects are.[17]

7. Interpretations and Pragmatism

How can we have the sort of access to mathematical states of affairs which is required for the understanding of mathematical propositions and for us to obtain mathematical knowledge? Does a pragmatist such as Peirce have the resources for answering these questions?

As was emphasized in section two, claims about the objects of propositions, terms and concepts have to be consistent with a plausible view of how we interpret those expressions. The interpretation has to identify the object, usually by being a further sign or thought which is determined by the same object, but which, usually, will develop the first sign by drawing inferences from the original and making use of background information. Peirce's pragmatist maxim stipulates that, if a concept or hypothesis is to have cognitive significance, then there must be circumstances in which the truth of the hypothesis, or the fact that the concept applies to something, can make a difference to what it is rational for us to do. The truth of the proposition must have practical consequences (see (Peirce 1878) and (Hookway 2004)). As we saw, one challenge to the metaphysical legitimacy of mathematical objects is that, since the objects of mathematical propositions are abstract, we cannot act upon them or interact with them. This suggests that facts about mathematical objects do not have practical consequences. Although I cannot provide a full explanation of Peirce's account of how we understand (and come to know) the propositions of mathematics, I shall identify three themes in Peirce's thought which may contribute to a satisfactory structuralist response to these issues.

First, as we saw in section six, the being of a number "merely consists in the truth of some proposition concerning a more primary substance" (Peirce 1903n:N4.162). There are intimate connections between propositions that involve reference to abstract objects such as numbers and propositions in which numerical expressions are applied in an adjectival way to collections, sets, and kinds. We arrive at terms for arithmetic objects by a step of hypostatic abstraction from statements which ascribe numbers adjectivally to things that are non-abstract. One way to trace the practical consequences of mathematical propositions is by moving in the other direction. I can interpret the thought that four plus seven is eleven by inferring that since I have eleven pounds (an adjectival use) then if I were to give you four pounds, I would be left with seven. Acquiring a piece of mathematical knowledge, in an appropriate context, can have important practical implications. So it is no surprise that mathematical knowledge has all sorts of practical consequences, once it is taken together with a mass of other empirical knowledge that we possess.

Second, even if the number series and its constituents are abstract, there will be instances of this structure which are concrete. For example, the series of counting vocables, or the series of numerals will constitute just such an instance. When we perform calculations or seek proofs in arithmetic, we can experiment upon diagrams (upon formulae, for example) and in doing this we experiment upon an instance of the abstract structure we are concerned with. Mathematical propositions have practical consequences simply because they have consequences for the results of our calculations and attempts at proof. We interact with the structure by interacting with one of its exemplifications. If this view required there to be a concrete exemplification of (for example) the entire structure of the natural numbers, it may follow that the truth of arithmetic depended upon there being infinitely many concrete objects, a view that is likely to be problematic. However it would be enough if we could make use of (and recognize) partial exemplifications of these structures, so long as we can explain how information obtained by studying some of these partial exemplifications can yield knowledge of the whole structure.

The third observation relates to the remarks about complex numbers at the end of he previous section. One of Peirce's earliest presentations of his pragmatism was in a review of a new edition of the writings of the philosopher George Berkeley (Peirce 1871). Peirce is concerned with Berkeley's rule for avoiding the risk of being deceived by words which have no clear meaning. He proposed that we should reject concepts and hypotheses unless we can associate clear ideas with them. In the mathematical case, this appeared to mean that we should abandon words and concepts for which no intelligible definition can be found. Peirce's pragmatism was offered as an alternative to Berkeley's rule:

> If such arguments had prevailed in mathematics (and Berkeley was equally strenuous in advocating them there) and if everything about negative quantities, the square root of *minus one*, and infinitesimals, had been excluded from the subject on the ground that we can form no idea of such things, the science would have been simplified no doubt, simplified by never advancing to the more difficult matters. (Peirce 1871:C8, pp. 33-34) (See also (Pycior 1995:135-136).)

Peirce's response is to propose an alternative rule of intellectual housekeeping: if "things fulfil the same function practically" then they are treated as different formulations of the same idea; and, we can assume, if something makes a difference practically, we should hold on to it even if we are currently ignorant of just what is going on. "If we ought to infer that [something] exists, if we only could frame the idea of it, why should we allow our mental incapacity to prevent us from adopting the proposition which logic requires?" We interpret things like the square root of minus one by using them in calculations and we gain the practical benefits of doing so. Beliefs about complex numbers have practical implications because they contribute to our ability to, for example, solve quadratic equations. We hope that, eventually, we will identify what the structures are in which they have a place, but we can use them and appreciate their value without having that knowledge.

I have argued that Peirce's philosophical view of mathematics has many significant resemblances to contemporary structuralism in the philosophy of mathematics. He was an *ante rem* structuralist who acknowledged that systems are metaphysically more fundamental than structures, numerical adjectives more fundamental than numbers. Exploring these relations casts illumination on some of his claims about the ontological status of mathematical objects such as the whole numbers.

Notes

1. I am very grateful to Bob Hale, Rosanna Keefe, David Liggins and Stewart Shapiro for comments on drafts, and discussions of related issues, that have led to substantial improvements in the paper.

2. We see Peirce facing this issue in his struggles with how to define "collections" or sets, recognizing that they are "*entia rationis*" (Peirce 1903e:N3.353) and "abstractions" (Peirce 1903n:N4.162). For illuminating discussion of this, see Moore's paper in this volume, pp. 334ff.

3. The idea that there is a tension between the most plausible account of what mathematics is about and the requirements for possessing knowledge in mathematics has become prominent through an important paper by Paul Benacerraf (Benacerraf 1973).

4. Peirce uses both 'real' and 'dynamical'. Sometimes he prefers 'dynamical' to 'real' because he wants to allow for fictional (and hence unreal) objects, but sometimes he links 'dynamical' to things that affect us though perception. Since it is his later usage, I shall use 'dynamical'.

5. See the discussion in (Short 2007:191).

6. After introducing these axioms, Peirce puts them to use in establishing theorems and corollaries.

7. The version given here shows the influence of Dedekind or Peano. Peirce's first axiomatisation was given in (Peirce 1881), seven or eight years before Dedekind's and Peano's. Paul Shields offers a detailed comparison of Peirce's early axiomatisation with these more familiar ones (Shields 1997:44-48).

8. One example of this is in (Peirce 1898d:C3, p. 357f). Stewart Shapiro has pointed out that Peirce's formulation has a meta-linguistic character and is reminiscent of Fraenkel's axiom of restriction in set theory.

9. The terminology used in the literature to mark this distinction is varied and often confusing. Parsons (Parsons 2004:57) distinguishes "eliminative" from "non-eliminative" structuralism, Dummett (Dummett 1991:295-297) contrasts "hard-headed" and "mystical" structuralism, and Hale (Hale 1996:125) speaks of "pure-structuralism" and "abstract-structuralism." I shall mostly rely upon Shapiro's vocabulary.

10. Macbride holds that structures are "universals" and, unlike Parsons as I understand him, ascribes the same view to Shapiro.

11. Words like 'object' and, as we shall soon see, 'nominalism' have to be handled very carefully when interpreting Peirce's writings. For many contemporary philosophers, an object is a kind of thing, an individual. Peirce understands the word much more broadly, so that any kind of expression can have objects; 'object' is closer to 'object of thought' than to 'thing'.

12. We also need to take account of the fact that Peirce's ideas developed over time and are mostly recorded in manuscripts so, perhaps, we should not expect them to be wholly consistent.

13. We shall find another reason for thinking that there can be such a distinction at the end of section six.

14. As he suggests in the final endnote of his paper, Moore's discussion of Peirce's understanding of collections in the final ten pages can be read as providing further evidence of a structuralist approach to matters of mathematical ontology.

15. The definition of 'symbolic logic' from which these quotes are taken was written in collaboration with H.B. Fine.

16. I am grateful to Bob Hale for pointing out that there is a potential problem here, one that I shall not investigate further in this paper. If there are infinitely many natural numbers and the truths about the numbers consist in the truth of propositions about more primary (presumably concrete) objects, it may be suggested that Peirce's account of mathematics requires the existence of infinitely many concrete objects. This would be an unfortunate consequence of the position. I am not aware of anywhere that Peirce actually considers this issue, and I shall not examine it further in this paper.

17. This case may still differ from those in which we discover that the dynamical object of our judgment that *that sheep is not moving* is, in fact, a bush. When we learn that a field can represent the complex numbers, do we *discover* that, when talking about *i*, we were always talking about a place in such a field?